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Test Groups and Effect Algebras

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A test group is a pair (G, T) where G is a partially ordered Abelian group and T is a generative antichain in its positive cone. It is shown here that effect algebras and algebraic test groups are coextensive, and a method for calculating the algebraic closure of a test group is developed. Some computational algorithms for studying finite effect algebras are introduced, and the problem of finding quotients of effect algebras is discussed.

Recently, it has come to light that virtually every mathematical structure that has been seriously proposed as a model for the propositions, properties, questions, events, effects, states, or observables affiliated with a physical system can be represented in terms of a partially ordered Abelian group, and that the highly developed theory of such groups can thus be brought to bear on problems arising in the experimental sciences (Greechie and Foulis, 1995). Although the full impact of the existence of a strong connection between physical systems and partially ordered Abelian groups has yet to be felt, and although some progress has already been made (Greechie *et al.*, 1995; Ravindran, 1995; Bennett and Foulis, 1995), many technical details have yet to be worked out before the connection can be fully exploited.

A large class of orthostructures, the so-called *interval effect algebras*, can be represented as order intervals $G^+[0, u]$ from 0 to an element u in the positive cone G^+ of a partially ordered Abelian group G (Bennett and Foulis, n.d.); however, in general, G enjoys none of the special properties that distinguish those classes of partially ordered Abelian groups that have been most intensively studied, e.g., lattice-ordered groups (Darnel, 1995) and inter-

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polation groups (Goodearl *et al.*, 1980; Goodearl, 1986; Murphy, 1992). The "test groups" introduced in this paper are often lattice ordered or interpolation groups and are related to effect algebras as test spaces (Foulis *et al.*, 1993) are to orthoalgebras (Foulis *et al.*, 1992). We study the articulation between test groups and effect algebras and show that much of the theory of test spaces can be extended to test groups.

Section 1 of this paper is devoted to a brief review of some relevant facts about partially ordered Abelian groups. The concept of a test group is introduced in Section 2. In Section 3, we show that algebraic test groups give rise to effect algebras and that every effect algebra arises in this way. We develop an iterative method to calculate the algebraic closure of a test group in Section 4. In Section 5, we prove that the universal group of an effect algebra E is a homomorphic image of an algebraic lattice-ordered test group corresponding to a set of generators for E. In Section 6, we sketch a basis for the construction of computational algorithms for dealing with finite effect algebras. We close the paper in Section 7 with a brief discussion of the problem of forming quotients of effect algebras.

1. PARTIALLY ORDERED ABELIAN GROUPS

Although most of the material in this section is well known in the theory of partially ordered Abelian groups (for instance, see Goodearl, 1986), we review it here for the convenience of the reader and to establish our notation and terminology. In what follows, Abelian groups are understood to be additively written. If G is an Abelian group and $M \subseteq G$, we define $\langle M \rangle$ to be the subgroup of G generated by M.

Let G be an Abelian group and let a, b, c denote elements of G. A subset C of G is called a *cone* if $C + C \subseteq C$ and $C \cap -C = \{0\}$. A cone C in G determines a partial order relation $\leq$ on G defined by $a \leq b$ iff $b - a \in C$, and $\leq$ is *translation invariant* in the sense that $a \leq b \Rightarrow a + c \leq b + c$. Conversely, if $\leq$ is a translation-invariant partial order on G, then $C := \{a \in G \mid 0 \leq a\}$ is a cone in G, called the *positive cone* determined by $\leq$. (The notation $:=$ means equal by definition.) In this way, a one-to-one correspondence is established between translation-invariant partial order relations $\leq$ on G and cones C in G.

A *partially ordered Abelian group* is an Abelian group G equipped with a translation-invariant partial order $\leq$. For such a group G, the positive cone determined by $\leq$ is denoted by G^+. The *standard positive cone* in the additive group $\mathbb{R}$ of real numbers is the set $\mathbb{R}^+$ of all real numbers that are nonnegative in the usual sense. *For the remainder of this section, let G be a partially ordered Abelian group.*

If H is a subgroup of G, then H is a partially ordered Abelian group in its own right under the restriction to H of the partial order $\leq$ on G. We refer to this as the *induced* partial order on H and to the corresponding positive cone $H^+ = H \cap G^+$ as the *induced positive cone*. For instance, the integers $\mathbb{Z}$ form a subgroup of $\mathbb{R}$ and the *standard positive cone* in $\mathbb{Z}$ is the induced positive cone $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+$.

A subgroup H of G is said to be *directed* (with respect to the induced partial order) iff for all $a, b \in H$, $\exists c \in H$ such that $a, b \leq c$.

Lemma 1.1. Let H be a subgroup of G and let $H^+ = H \cap G^+$. Then the following conditions are mutually equivalent: (i) H is directed; (ii) $H = H^+ - H^+$; (iii) $H = \langle H^+ \rangle$; (iv) $\exists M \subseteq G^+$ with $H = \langle M \rangle$.

Proof. (i) $\Rightarrow$ (ii): Assume (i) and let $h \in H$. Then $\exists x \in H$ with $0, h \leq x$; hence, $x \in H \cap G^+ = H^+$, $y := x - h \in H \cap G^+ = H^+$, and $h = x - y$. Obviously, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i): Assume (iv) and let $a, b \in H$. Then $\exists x, y, z, w \in M \cap H^+$ such that $a = x - y \leq x \in H$ and $b = z - w \leq z \in H$, and it follows that $a, b \leq c := x + z \in H$. ■

Corollary 1.2. $G = \langle G^+ \rangle$ iff G is directed.

A subgroup H of G is called *order-convex* iff, for all $a, b \in G$, $0 \leq a \leq b \in H \Rightarrow a \in H$. In the theory of partially ordered Abelian groups, an order-convex directed subgroup H of G is referred to as an *ideal*; however, we avoid using this terminology because of possible confusion with the notion of an ideal in an effect algebra. (See Section 7 below.) If $H^+ = H \cap G^+ = \{0\}$, then H is said to be *trivially order-convex*.

Theorem 1.3. Let H be a subgroup of G. Then the following conditions are mutually equivalent:

(i) H is order-convex.

(ii) $a, c \in H$, $b \in G$, $a \leq b \leq c \Rightarrow b \in H$.

(iii) If Q is an Abelian group and η: $G \to Q$ is a group homomorphism with $\ker(\eta) = H$, then $\eta(G^+)$ is a cone in Q.

(iv) There is a partially ordered Abelian group Q and a group homomorphism η: $G \to Q$ such that $\eta(G^+) \subseteq Q^+$ and $H = \ker(\eta)$.

Proof. (i) $\Rightarrow$ (ii): Assume (i) and the hypotheses of (ii). Then $0 \leq b - a \leq c - a \in H$, so $b - a \in H$, and it follows that $b \in H$. Obviously, (ii) $\Rightarrow$ (i), so (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (iii): Assume (i) and the hypotheses of (iii). Obviously, $\eta(G^+) + \eta(G^+) \subseteq \eta(G^+)$ and $0 \in \eta(G^+)$. Suppose $q, -q \in \eta(G^+)$. Then $\exists a, b \in G^+$, $q = \eta(a)$, $-q = \eta(b)$. Then $0 \leq a \leq a + b$ and $\eta(a + b) = q - q = 0$, so $a + b \in \ker(\eta) = H$, and it follows that $a \in H = \ker(\eta)$, so $q = \eta(a) = 0$.

(iii) $\Rightarrow$ (iv): Assume (iii), let $Q = G/H$, and let η: $G \to Q$ be the natural group epimorphism. By (iii), Q can be organized into a partially ordered Abelian group with positive cone $Q^+ := \eta(G^+)$.

(iv) $\Rightarrow$ (i): Assume (iv) and let $a, b \in G$, $0 \le a \le b \in H$. Then $a \in G^+$, so $\eta(a) \in G^+$. Also, $b - a \in G^+$ and $b \in H = \ker(\eta)$, so $-\eta(a) = \eta(b - a) \in G^+$, and it follows that $\eta(a) = 0$, so $a \in \ker(\eta) = H$. ■

Let H be an order-convex subgroup of G and let η: $G \to G/H$ be the natural homomorphism. Then, by Part (ii) of Theorem 1.3, G/H can be organized into a partially ordered Abelian group with positive cone $(G/H)^+ := \eta(G^+)$. Unless one makes an explicit stipulation to the contrary, it is always understood that G/H is partially ordered in this way. Obviously, if G is directed, so is G/H.

Definition 1.4. If M is a subset of G, we define $\mathrm{ocs}(M)$ to be the intersection of all order-convex subgroups of G that contain M. We also define $\mathrm{ssg}(M)$ to be the subsemigroup of G consisting of 0 and all sums of finite sequences of elements in M.

The intersection of order-convex subgroups of G is again an order-convex subgroup of G, so $\mathrm{ocs}(M)$ is the smallest order-convex subgroup of G that contains M.

Lemma 1.5. Let $M \subseteq G^+$. Then:
(i) $\mathrm{ocs}(M) = \mathrm{ocs}(\mathrm{ssg}(M)) = \{h \in G \mid \exists y \in \mathrm{ssg}(M), -y \le h \le y\}$.
(ii) $\mathrm{ocs}(M)$ is a directed order-convex subgroup of G.

Proof. (i) Obviously $\mathrm{ssg}(M) \subseteq G^+$ and $\mathrm{ocs}(M) = \mathrm{ocs}(\mathrm{ssg}(M))$. Let $H := \{h \in G \mid \exists y \in \mathrm{ssg}(M), -y \le h \le y\}$ and let $h, k \in H$. Then $\exists y, z \in \mathrm{ssg}(M)$ with $-y \le h \le y$ and $-z \le k \le z$. Thus, $-z \le -k \le z$, and $w := y + z \in \mathrm{ssg}(M)$ with $-w \le h - k \le w$, and it follows that H is a subgroup of G with $M \subseteq \mathrm{ssg}(M) \subseteq H$. Suppose $g \in G$ and $0 \le g \le h \in H$. Then $\exists y \in \mathrm{ssg}(M) \subseteq G^+$ with $-y \le h \le y$, so $-y \le 0 \le g \le y$, and it follows that $g \in H$. Therefore, H is an order-convex subgroup of G. That any order-convex subgroup of G that contains $\mathrm{ssg}(M)$ must contain H is clear.

(ii) Let $h, k \in \mathrm{ocs}(M)$. By Part (i), $\exists y, z \in \mathrm{ssg}(M) \subseteq G^+$ with $-y \le h \le y$ and $-z \le k \le z$, and it follows that $h, k \le y + z \in \mathrm{ocs}(M)$. ■

The partially ordered Abelian group G is said to be *lattice-ordered* iff the partially ordered set $(G, \le)$ is a lattice. If G is lattice-ordered and H is an order-convex directed subgroup of G, then, as a partially ordered set, H is clearly a sublattice of G, so H is lattice-ordered. If G is lattice-ordered and $x_1, x_2, y_1, y_2 \in G$ with $x_i \le y_j$ for all i, j, then any element $z \in G$ with

$x_1 \vee x_2 \leq z \leq y_1 \wedge y_2$ satisfies the conditions $x_i \leq z \leq y_j$ for all i, j. Thus, if G is lattice-ordered, it satisfies the condition in the following definition.

Definition 1.6. G is an *interpolation group* (Goodearl, 1986) iff x_1, x_2, y_1, $y_2 \in G$ with $x_i \leq y_j$ for all $i, j \Rightarrow \exists z \in G$, $x_i \leq z \leq y_j$ for all i, j.

If G is an interpolation group and H is an order-convex directed subgroup of G, then both H and G/H are interpolation groups (Goodearl, 1986, Proposition 2.3).

Definition 1.7. Let $T \subseteq G^+$.
(i) $G^+[0, T] := \{p \in G^+ \mid \exists t \in T, p \leq t\}$.
(ii) T is *generative* iff $G^+ = \text{ssg}(G^+[0, T])$ and $G = \langle G^+ \rangle$.

Note that we speak of a generative set in G only when G is directed. Thus, T is generative iff $G^+[0, T]$ generates G^+ as a semigroup, and G^+, in turn, generates G as a group.

Lemma 1.8. Let $T \subseteq G^+$ and consider the following conditions:
(i) T is generative.
(ii) $g \in G \Rightarrow \exists c \in \text{ssg}(T)$, $g \leq c$.
(iii) $\text{ocs}(T) = G$.
(iv) G is directed.
Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv). If G is an interpolation group, then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Proof. To prove (i) $\Rightarrow$ (ii), assume (i) and suppose $g \in G$. Since G is directed, $\exists a, b \in G^+$ with $g = a - b$ and, since T is generative, there are finite sequences $p_1, p_2, p_3, \ldots, p_n$ in G^+ and $t_1, t_2, t_3, \ldots, t_n$ in T with $p_i \leq t_i$ for $i = 1, 2, 3, \ldots, n$ and $a = \Sigma_i\, p_i$. Let $c := \Sigma_i\, t_i$. Then, $a = \Sigma_i\, p_i \leq \Sigma_i\, t_i = c \leq b + c$, whence $g = a - b \leq c \in \text{ssg}(T)$. That (iii) $\Rightarrow$ (ii) follows directly from Part (i) of Lemma 1.5. To prove (ii) $\Rightarrow$ (iii), assume (ii) and let $g \in G$. Choose $c, d \in \text{ssg}(T)$ with $g \leq c$ and $-g \leq d$ and let $y = c + d \in \text{ssg}(T) \subseteq G^+$. Then $-y \leq g \leq y$, and it follows that $g \in \text{ocs}(T)$. That (iii) $\Rightarrow$ (iv) follows from Part (ii) of Lemma 1.5.

Finally, suppose G is an interpolation group and that it satisfies Condition (ii). Since (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), we have only to prove that T is generative. Thus, suppose $g \in G^+$. By (ii), $\exists t_1, t_2, t_3, \ldots, t_n \in T$ such that $g \leq \Sigma_i\, t_i$. By Proposition 2.2 in Goodearl (1986), $\exists p_1, p_2, p_3, \ldots, p_n \in G^+$ such that $g = \Sigma_i\, p_i$ and $p_i \leq t_i$ for $i = 1, 2, 3, \ldots, n$; and it follows that $g \in \text{ssg}(G^+[0, T])$. ■

An element $u \in G^+$ is said to be *generative* iff $\{u\}$ is a generative set (Bennett and Foulis, n.d.), and it is called an *order-unit* (Goodearl *et al.*, 1980) iff, for every $g \in G$, $\exists n \in \mathbb{Z}^+$, $g \leq nu$. By Part (ii) of Lemma 1.8, a generative element is automatically an order-unit. However, unless G is an interpolation group, not every order-unit need be generative. For instance, if $G = \mathbb{Z}$ as an Abelian group with the nonstandard cone $G^+ := \mathbb{Z}^+\backslash\{1\}$, then G is directed and 2 is an order-unit in G, but it is not generative.

Lemma 1.9. Let $T \subseteq G^+$ be generative and let H be an order-convex directed subgroup of G. Then $H = \langle H \cap G^+[0, T]\rangle$.

Proof. By Lemma 1.1, $H = \langle H \cap G^+\rangle$, so it will be sufficient to prove that $H \cap G^+ \subseteq \langle H \cap G^+[0, T]\rangle$. Let $h \in H \cap G^+$. Since T is generative, there are finite sequences $h_1, h_2, \ldots, h_n \in G^+$ and $t_1, t_2, \ldots, t_n \in T$ with $h_i \le t_i$ for $i = 1, 2, 3, \ldots, n$ and $h = \Sigma_i h_i$. For each $j = 1, 2, \ldots, n$, we have $0 \le h_j \le \Sigma_i h_i = h \in H$ and, owing to the order-convexity of H, we have $h_j \in H \cap G^+$. ■

If X is a nonempty set, then $\mathbb{Z}^X := \{f \mid f: X \to \mathbb{Z}\}$ is a lattice-ordered Abelian group under pointwise addition and with the *standard positive cone* $(\mathbb{Z}^+)^X$. If $f \in \mathbb{Z}^X$, the *support* of f is the set $\operatorname{supp}(f) := \{x \in X \mid f(x) \neq 0\}$. The subgroup of $\mathbb{Z}^X$ consisting of all functions with finite support is denoted by $\mathbb{Z}^{[X]}$ and the induced positive cone in $\mathbb{Z}^{[X]}$ is $(\mathbb{Z}^+)^{[X]}$. Evidently, $\mathbb{Z}^{[X]}$ is an order-convex directed subgroup of $\mathbb{Z}^X$ and therefore is lattice-ordered. For the remainder of this paper, we always understand that $\mathbb{Z}^X$ and $\mathbb{Z}^{[X]}$ are thus organized into lattice-ordered groups.

The partially ordered Abelian group G is called a *simplicial group* (Goodearl, 1986, p. 47) iff it is isomorphic (as a partially ordered group) to $\mathbb{Z}^X$ for some finite set X. A partially ordered Abelian group G is simplicial iff it is an interpolation group with an order unit and (as a partially ordered set) G^+ satisfies the descending chain condition (Goodearl, 1986, Corollary 3.14).

Theorem 1.10. Let X be a set. Then:

(i) If $Y \subseteq X$ and $H = \{f \in \mathbb{Z}^{[X]} \mid f(y) = 0 \text{ for all } y \in Y\}$, then H is a directed order convex subgroup of $\mathbb{Z}^{[X]}$.

(ii) If X is a finite set and H is a subgroup of $\mathbb{Z}^X$, then H is directed and order convex iff it has the form in Part (i) for some subset Y of X.

Proof. Part (i) is obvious. (ii) By Goodearl (1986), Proposition 3.8, if X is finite, then H is directed and order-convex iff there is a subset W of X such that H is generated by all characteristic set functions $\chi_{\{w\}}$ with $w \in W$. Let $Y = X\backslash W$. ■

A subset T of G is called an *antichain* iff no two distinct elements of T are comparable with respect to the partial order $\le$. Thus, $T \subseteq G$ is an antichain iff $s, t \in T$ with $s \le t \Rightarrow s = t$.

Lemma 1.11. If G is a simplicial group, then every antichain in G^+ is finite.

Proof. See Perles (1963). ∎

2. TEST GROUPS

The idea of a test group, which we now introduce, unifies a number of different concepts which have been studied in connection with effect algebras and difference posets. This part of our work is related to, and partially motivated by, the D-test spaces of Dvurečenskij and Pulmannová (1994b).

Definition 2.1. A *test group* is a pair (G, T) consisting of a partially ordered Abelian group G with positive cone G^+ and a nonempty generative antichain $T \subseteq G^+$. Elements $t \in T$ are called *tests* and elements in $G^+[0, T]$ are called *T-events*, or simply *events* if T is understood.

The following example shows how test groups generalize interval effect algebras (Bennett and Foulis, n.d.).

Example 2.2. $G^+[0, u]$ is an interval effect algebra with ambient group G and unit u iff $(G, \{u\})$ is a test group.

The next example and definition are basic, as they show how lattice-ordered test groups typically arise from effect algebras. For the definition and elementary facts about effect algebras see Foulis and Bennett (1994). If E is an effect algebra, a subset X of $E\setminus\{0\}$ is called a set of *generators* for E iff every $a \in E$ can be written as a finite orthocombination $a = \oplus_{x\in X} f(x)x$, where $f \in (\mathbb{Z}^+)^{[X]}$.

Example 2.3. Let E be an effect algebra with unit u, let $X \subseteq E\setminus\{0\}$ be a set of generators for E and let

$$T = \{t \in (\mathbb{Z}^+)^{[X]} \mid u = \oplus_{x\in X} t(x)x\}$$

Then $(\mathbb{Z}^{[X]}, T)$ is a test group and the T-events are the functions $f \in (\mathbb{Z}^+)^{[X]}$ for which $\oplus_{x\in X} f(x)x$ is defined in E. If X is finite, then $\mathbb{Z}^{[X]} = \mathbb{Z}^X$ is a simplicial group and, by Lemma 1.11, T is necessarily finite, hence E itself is finite.

If the effect algebra E is finite, or more generally, satisfies the descending chain condition, then one can choose the set X of atoms in E as a set of generators. If no obvious set of generators presents itself, one can always choose $X = E\setminus\{0\}$.

Definition 2.4. If X is a set and $\mathbb{Z}^{[X]}$ is partially ordered by the standard positive cone $(\mathbb{Z}^+)^{[X]}$, a test group of the form $(\mathbb{Z}^{[X]}, T)$ is called a *multiplicity group* provided that, for every $x \in X$, $\exists t \in T$, $t(x) \neq 0$. If E is an effect algebra with unit u and $X \subseteq E\setminus\{0\}$ is a set of generators for E, a function

$t \in \mathbb{Z}^{[X]}$ is called a *multiplicity function* for E with respect to X iff $u = \oplus_{x \in X} t(x)x$. If T is the set of all multiplicity functions for E with respect to X, the test group $(\mathbb{Z}^{[X]}, T)$ is called the *multiplicity group* for E with respect to X.

The following example shows how test spaces (Foulis *et al.*, 1993) give rise to lattice-ordered test groups.

Example 2.5. Let $(X, \mathcal{F})$ be a test space and, for each test $E \in \mathcal{F}$, let χ_E be the characteristic set function of E. Let $T := \{\chi_E | E \in \mathcal{F}\}$ and let G be the order-convex subgroup of $\mathbb{Z}^X$ generated by T. Then (G, T) is a test group for which the T-events are the characteristic set functions of the events for $(X, \mathcal{F})$.

By analogy with Example 2.5, the D-test spaces of Dvurečenskij and Pulmannová (1994a,b) also give rise to lattice-ordered test groups.

Example 2.6. Let $(X, \mathcal{F})$ be a D-test space and, for each $E \in \mathcal{F}$ and all $x \in X$, let $t_E(x)$ be the (finite) cardinal number of the set $E^{-1}(x)$. Let $T := \{t_E | E \in \mathcal{F}\} \subseteq \mathbb{Z}^X$, and let $G := \mathrm{ocs}(T)$. Then (G, T) is a test group and the T-events may be identified with the events for $(X, \mathcal{F})$.

For the remainder of this paper, *we assume that (G, T) is a test group with $G^+[0, T]$ as its set of T-events.* The proof of the following lemma is straightforward.

Lemma 2.7. If $a \in G^+$, then the following conditions are mutually equivalent:

(i) $a \in G^+[0, T]$.
(ii) $\exists b \in G^+, a + b \in T$.
(iii) $\exists b \in G^+[0, T], a \leq b$.

Definition 2.8. Let $a, b, c \in G^+[0, T]$.

(i) a is *orthogonal* to c, in symbols, $a \perp c$, iff $a + c \in G^+[0, T]$.
(ii) a and c are *supplements* iff $a + c \in T$.
(iii) a and b are *perspective*, in symbols $a \sim b$, iff a and b share a common supplement c. Such a common supplement c is called an *axis* for the perspectivity $a \sim b$.

As we now show, most of the facts established in Foulis *et al.* (1993) for perspectivity in a test space can be generalized to test groups. For a related study of perspectivity in partial Abelian semigroups, see Pulmannová and Wilce (1994) and Wilce (1995).

Lemma 2.9. Let $a, b, c \in G^+[0, T]$ and let $t, u \in T$. Then:

(i) $t \sim u$ with axis 0.
(ii) $a \perp c$ and $c \sim (a + c) \Rightarrow a = 0$.

(iii) $a \sim 0 \Rightarrow a = 0$.
(iv) $a \leq b$ and $b \perp c \Rightarrow a \perp c$.
(v) $a \sim t \Leftrightarrow a \in T$.

Proof. Part (i) is obvious. To prove Part (ii), assume the hypotheses and let d be an axis for $c \sim (a + c)$. Then, $c + d$, $a + c + d \in T$ with $c + d \leq a + c + d$, and it follows that $c + d = a + c + d$, whence $a = 0$. Part (iii) follows from Part (ii) with $c = 0$. From the hypotheses of Part (iv) we have $0 \leq a + c \leq b + c \in G^+[0, T]$, so $a + c \in G^+[0, T]$ and therefore $a \perp c$. In Part (v), the implication $\Leftarrow$ follows from Part (i). To prove the converse implication in Part (v), note that if d is an axis for $a \sim t$, then t, $t + d \in T$ with $t \leq t + d$, whence $t = t + d$, so $d = 0$ and $a = a + d \in T$. ■

Lemma 2.10 (Perspectivity Cancellation Law). Let $a, b, c \in G^+[0, T]$ with $a \perp c$ and $b \perp c$. Then $a + c \sim b + c \Rightarrow a \sim b$.

Proof. By hypothesis, $\exists d \in G^+[0, T]$ such that $a + c + d$ and $b + c + d \in T$. Thus a is perspective to b with axis $c + d$. ■

Lemma 2.11 (Additivity Lemma). Suppose that $\sim$ is a transitive relation and let $a, b, c \in G^+[0, T]$ with $a \perp b$ and $a \perp c$. Then $b \sim c \Rightarrow a + b \sim a + c$.

Proof. Let d be an axis for $b \sim c$ and choose $p, q \in G^+[0, T]$ such that $a + b + p$, $a + c + q \in T$. Then $a + p \sim d$ with axis b and $d \sim a + q$ with axis c. Since $\sim$ is transitive, $a + p \sim a + q$; hence, $p \sim q$ by Lemma 2.10. Let r be an axis for $p \sim q$, so that $a + b \sim r$ with axis p. Furthermore, $r \sim a + c$ with axis q; hence, $a + b \sim a + c$ by the transitivity of $\sim$. ■

We now introduce a reflexive and transitive relation $\lesssim$ on $G^+[0, T]$ that extends both $\leq$ and $\sim$.

Definition 2.12. If $a, b \in G^+[0, T]$, define $a \lesssim b$ iff there is a finite sequence $c_0, c_1, c_2, \ldots, c_n$ in $G^+[0, T]$ such that $a = c_0$, $b = c_n$, and, for each $i = 1, 2, 3, \ldots, n$, one of the conditions $c_{i-1} \leq c_i$ or $c_{i-1} \sim c_i$ holds.

Lemma 2.13. If $a \in G^+[0, T]$ and $t \in T$, then (i) $0 \lesssim a \lesssim t$, (ii) $a \lesssim 0 \Rightarrow a = 0$, and (iii) $t \lesssim a \Rightarrow a \in T$.

Proof. Lemma 2.9 and induction. ■

Lemma 2.14. If $a, b, a', b' \in G^+[0, T]$, $a + a' \in T$, $b + b' \in T$, and $a \lesssim b$, then $b' \lesssim a'$.

Proof. Assume the hypotheses. By induction, it will be enough to prove that $b' \lesssim a'$ for the two special cases $a \leq b$ and $a \sim b$. If $a \leq b$, let $c = b - a$ and observe that $b' \leq (b' + c) \sim a'$ with axis a, so $b' \lesssim a'$. If $a \sim b$ with axis d, then $b' \sim d$ with axis b, $d \sim a'$ with axis a, and again $b' \lesssim a'$. ■

3. ALGEBRAIC SETS OF TESTS

The notion of an algebraic test group as given in the following definition generalizes the idea of an algebraic test space (Foulis *et al.*, 1993).

Definition 3.1. We say that the test group (G, T) is *algebraic* iff perspectivity preserves orthogonality; that is, for all $a, b, c \in G^+[0, T]$,

$$a \sim b \text{ and } b \perp c \Rightarrow a \perp c$$

We also express the idea that (G, T) is algebraic by saying that T is *algebraic in* G or, if G is understood, simply that T is *algebraic.*

Theorem 3.2. The following conditions are mutually equivalent:

(i) T is algebraic.

(ii) If $a, b, c \in G^+[0, T]$, $a \lesssim b$, and $b \perp c$, then $a \perp c$.

(iii) If $a, b, c \in G^+[0, T]$, $a \sim b$, and c is a supplement of a, then c is a supplement of b.

Proof. That (i) $\Rightarrow$ (ii) follows immediately from Definition 3.1, Part (iv) of Lemma 2.9, and induction. That (ii) $\Rightarrow$ (i) is obvious. To prove (i) $\Rightarrow$ (iii), assume (i) and the hypothesis of (iii) and let d be an axis for $a \sim b$. Then $d \sim c$ with axis a. Since $b \sim a$ and $a \perp c$, it follows from (i) that $b \perp c$. Let p be a supplement of $b + c$, so that $(c + p) \sim d$ with axis b. Therefore, since $a \perp d$, it follows from (i) that $a \perp (c + p)$; hence, $\exists t \in T$ with $a + c + p \leq t$. But then, $a + c \in T$ with $a + c \leq t$, and it follows that $a + c = t$, so $p = 0$ and c is a supplement of b. To prove (iii) $\Rightarrow$ (i), assume (iii) and suppose that $a \sim b$, and $b \perp c$. Let q be a supplement of $b + c$. Then $c + q$ is a supplement of b, so $c + q$ is a supplement of a, and it follows that $c \perp a$. ■

Corollary 3.3. Let T be algebraic and let $a, b \in G^+[0, T]$. Then:

(i) $a \lesssim b \Leftrightarrow \exists c \in G^+[0, T]$ with $a \leq c$ and $c \sim b$.

(ii) $\sim$ is an equivalence relation on $G^+[0, T]$.

(iii) $a \lesssim b$ and $b \lesssim a \Leftrightarrow a \sim b$.

Proof. The implication $\Leftarrow$ in Part (i) is obvious. To prove the converse implication, suppose $a \lesssim b$ and let d be a supplement of b. Then $a \perp d$ by Part (ii) of Theorem 3.2. Let p be a supplement of $a + d$ and let $c = a + p$, so that $c + d \in T$. Then $a \leq c$ and $c \sim b$ with axis d. To prove Part (ii), it suffices to show that $\sim$ is transitive. Thus, suppose $a, b, d \in S$ with $a \sim b$ and $b \sim d$, and let c be an axis for $b \sim d$. Then c is a supplement of b, and it follows from Part (iii) of Theorem 3.2 that c is a supplement of a; hence, $a \sim d$ with axis c. In Part (iii), the implication $\Leftarrow$ is obvious. To prove the converse, assume that $a \lesssim b$ and $b \lesssim a$. By Part (i), $\exists a_1, b_1$ with $a \perp a_1$, $b \perp b_1$, $(a + a_1) \sim b$, and $(b + b_1) \sim a$. Since $b \perp b_1$ and T is algebraic, we have $(a + a_1) \perp b_1$. Thus, by Part (ii) and Lemma 2.11, $(a + a_1) + b_1 \sim b + b_1 \sim a$, and therefore $a_1 + b_1 = 0$ by Part (ii) of Lemma 2.9. Consequently, $a_1 = b_1 = 0$, so $a \sim b$. ■

Theorem 3.4. (Additivity Theorem). Suppose T is algebraic and let a_1, a_2, b_1, $b_2 \in G^+[0, T]$ with $a_1 \sim a_2$, $b_1 \sim b_2$, and $a_1 \perp b_1$. Then $a_2 \perp b_2$ and $a_1 + b_1 \sim a_2 + b_2$.

Proof. That $a_2 \perp b_2$ follows from two applications of the condition in Definition 3.1. By Part (ii) of Corollary 3.3, $\sim$ is transitive. Thus, using Lemma 2.11 twice, we have $a_1 + b_1 \sim a_1 + b_2 \sim a_2 + b_2$. ■

If T is algebraic, the *perspectivity equivalence classes* in $G^+[0, T]$ can be organized into an effect algebra as follows.

Definition 3.5. Suppose T is algebraic. For each $a \in G^+[0, T]$, define $\pi(a) := \{b \in G^+[0, T] \mid a \sim b\}$ and define $\Pi(G, T) := \{\pi(a) \mid a \in G^+[0, T]\}$. If $\pi(a), \pi(b) \in \Pi(G, T)$, define $\pi(a) \oplus \pi(b)$ iff $a \perp b$, in which case $\pi(a) \oplus \pi(b) := \pi(a + b)$. By Theorem 3.4, $\oplus$ is well defined.

The following theorem is an immediate consequence of the preceding results.

Theorem 3.6. If T is algebraic, then $\Pi(G, T)$ is an effect algebra with zero element $\pi(0)$ and with unit $u := \pi(t)$, where t is any element of T. Furthermore, for $a, b \in G^+[0, T]$, $a \lesssim b \Leftrightarrow \pi(a) \leq \pi(b)$ in $\Pi(G, T)$.

In Example 2.2, $\{u\}$ is algebraic and $\Pi(G, \{u\})$ can be identified with the interval effect algebra $G^+[0, u]$. A multiplicity group $(\mathbb{Z}^{[X]}, T)$ for an effect algebra E is automatically algebraic and $\Pi(\mathbb{Z}^{[X]}, T)$ is isomorphic to E under the well-defined mapping $\pi(f) \mapsto \oplus_{x \in X} f(x)x$. Consequently, *every effect algebra can be represented as* $\Pi(G, T)$ *for an algebraic lattice-ordered test group*. In Examples 2.5 and 2.6, T is algebraic iff $(X, \mathcal{F})$ is algebraic in the sense of Foulis *et al.* (1993) and Dvurečenskij and Pulmannová (1994b), respectively.

4. THE ALGEBRAIC CLOSURE OF T

If a set of tests T fails to be algebraic, it is because there are T-events a, b, c, d such that $a + c$, $c + b$, and $b + d$ belong to T, but $a + d$ does not. In other words, T fails to be algebraic just because it fails to be "large enough." This suggests that it might be possible to enlarge a nonalgebraic T to an algebraic set of tests. Of course, enlarging T means enlarging the set of T-events, and repeated enlargements might be necessary before one arrives (if at all) at an algebraic set of tests.

Note that, if $T \subseteq S \subseteq G^+$, then S is automatically generative; hence, (G, S) is a test group iff S is an antichain.

Definition 4.1. The test group (G, T) is said to be *prealgebraic* iff there is an antichain $S \subseteq G^+$ such that $T \subseteq S$ and (G, S) is an algebraic test group.

We also express the idea that (G, T) is prealgebraic by saying that T is *prealgebraic in G* or, if G is understood, simply that T is *prealgebraic.*

If $T \subseteq S_\iota \subseteq G^+$ and S_ι is an algebraic antichain for all ι, it is clear that $T \subseteq \bigcap_\iota S_\iota$ and $\bigcap_\iota S_\iota$ is an algebraic antichain in G^+. Thus, we have the following definition and theorem.

Definition 4.2. If T is prealgebraic in G, then $T^\frown$, called the *algebraic closure of T in G,* is the set-theoretic intersection of all algebraic antichains $S \subseteq G^+$ such that $T \subseteq S$. We also refer to $(G, T^\frown)$ as the *algebraic closure* of (G, T).

If G is understood and T is prealgebraic, we may refer to $T^\frown$ simply as the *algebraic closure* of T. Definition 4.2 has the following obvious consequence.

Theorem 4.3. If T is prealgebraic, then $T^\frown$ is algebraic and if S is an algebraic antichain with $T \subseteq S \subseteq G^+$, then $T^\frown \subseteq S$.

We now proceed to show that, if T is prealgebraic and G is an interpolation group, then it is possible to form the algebraic closure $T^\frown$ inductively by successively adjoining elements to T.

Definition 4.4. If $S \subseteq G^+$, we now define the *derived set* $\mathfrak{D}(S)$ by $\mathfrak{D}(S) := \{t + v - s \mid s, t, v \in S, s \leq t + v\}$ and, by induction, $\mathfrak{D}^0(S) = S$ and $\mathfrak{D}^{n+1}(S) = \mathfrak{D}(\mathfrak{D}^n(S))$ for $n = 1, 2, 3, \ldots$.

Note that $S \subseteq \mathfrak{D}^n(S) \subseteq \mathfrak{D}^{n+1}(S)$ for all $S \subseteq G^+$ and all $n = 0, 1, 2, \ldots$.

Theorem 4.5. If $\mathfrak{D}(T) \subseteq T$, then T is algebraic. Conversely, if T is algebraic and G is an interpolation group, then $\mathfrak{D}(T) = T$.

Proof. Suppose $\mathfrak{D}(T) \subseteq T$ and let $a, b, c \in G^+[0, T]$ with $a \sim b$ and $b + c \in T$. By Theorem 3.2, it will be sufficient to prove that $a + c \in T$. Let d be an axis for $a \sim b$ and put $t = a + d$, $s = d + b$, $v = b + c$, noting that $s, t, v \in T$ and $s = d + b \leq (a + d) + (b + c) = t + v$, whence $a + c = (a + d) + (b + c) - (d + b) = t + v - s \in \mathfrak{D}(T) \subseteq T$.

Conversely, suppose T is algebraic and G is an interpolation group. Since $T \subseteq \mathfrak{D}(T)$, we have to prove $\mathfrak{D}(T) \subseteq T$. Let $s, t, v \in T$ with $s \leq t + v$, noting that $0, s - v \leq s, t$. Hence, $\exists d \in G$ with $0, s - v \leq d \leq s, t$. Because $0 \leq d \leq t$, we have $d \in G^+[0, T]$. Let $a = t - d$, $b = s - d$, and $c = v + d - s$. Then $0 \leq a \leq t$ and $0 \leq b \leq s$, so $a, b \in G^+[0, T]$. Also $a + d = t \in T$ and $b + d = s \in T$, so $a \sim b$ with axis d. Furthermore,

$s - v \leq d$ implies that $0 \leq v + d - s = c$ and, since $b + c = (s - d) + (v + d - s) = v \in T$, it follows that c is a supplement of b. Consequently, the fact that T is algebraic implies that

$$t + v - s = (t - d) + (v + d - s) = a + c \in T \quad \blacksquare$$

Theorem 4.6. If $\mathfrak{D}^n(T)$ is an antichain for all $n = 0, 1, 2, \ldots$, then T is prealgebraic and $T^{\sim} \subseteq \bigcup_n \mathfrak{D}^n(T)$. Conversely, if G is an interpolation group and T is prealgebraic, then $\mathfrak{D}^n(T)$ is an antichain for all $n = 0, 1, 2, \ldots$, and $T^{\sim} = \bigcup_n \mathfrak{D}^n(T)$.

Proof. Let $S = \bigcup_n \mathfrak{D}^n(T)$, noting that $T \subseteq S$ and $\mathfrak{D}(S) = S$. Suppose that $\mathfrak{D}^n(T)$ is an antichain for all n. Then, owing to the fact that $T \subseteq \mathfrak{D}^n(T) \subseteq \mathfrak{D}^{n+1}(T)$ for all n, S is an antichain that contains T and S is algebraic by Theorem 4.5; hence T is prealgebraic.

Conversely, suppose G is an interpolation group and T is prealgebraic. Then $T^{\sim}$ is algebraic, $T \subseteq T^{\sim}$, and $T^{\sim} = \mathfrak{D}(T^{\sim})$ by Theorem 4.5. Therefore, $\mathfrak{D}^n(T) \subseteq \mathfrak{D}^n(T^{\sim}) = T^{\sim}$ for all $n = 1, 2, 3, \ldots$; hence, since $T^{\sim}$ is an antichain, so is $\mathfrak{D}^n(T)$. Furthermore, $S \subseteq T^{\sim}$ and, since $\mathfrak{D}(S) = S$, it follows from Theorem 4.5 that S is algebraic, so $T^{\sim} \subseteq S$, and therefore $T^{\sim} = S$. $\blacksquare$

Theorem 4.7. Suppose (Q, U) is an algebraic test group and that $\psi\colon G \to Q$ is a group homomorphism such that $\psi(G^+) \subseteq Q^+$, $\psi(T) \subseteq U$, and $\ker(\psi) \cap G^+ = \{0\}$. Then (G, T) is prealgebraic and $\psi(T^{\sim}) \subseteq U$.

Proof. Let $S := \{s \in G^+ \mid \psi(s) \in U\}$. Suppose $p, q \in S$ with $p \leq q$. Since $\psi(G^+) \subseteq Q^+$, we have $\psi(p) \leq \psi(q)$; hence, since $\psi(p), \psi(q) \in U$, we have $\psi(p) = \psi(q)$. Therefore, $q - p \in \ker(\psi) \cap G^+ = \{0\}$, so $p = q$, proving that S is an antichain in G^+. Since $T \subseteq S$, it follows that T is prealgebraic and $T^{\sim} \subseteq S$. Therefore, $\psi(T^{\sim}) \subseteq \psi(S) \subseteq U$. $\blacksquare$

5. THE UNIVERSAL GROUP

Given an effect algebra E and a set of generators $X \subseteq E\setminus\{0\}$, one can form the corresponding algebraic multiplicity group $(\mathbb{Z}^{[X]}, T)$ as in Definition 2.4, thus realizing E, up to isomorphism, as $\Pi(\mathbb{Z}^{[X]}, T)$. Conversely, one can *start with* a set X and an algebraic multiplicity group $(\mathbb{Z}^{[X]}, T)$ and form the corresponding effect algebra $E := \Pi(\mathbb{Z}^{[X]}, T)$, and this will be our point of view in the present section. As a matter of fact, this gives one of the most effective methods for specifying finite effect algebras. Thus, we fix the following notation for the remainder of this section.

Notation 5.1. X is a set, $G := \mathbb{Z}^{[X]}$ with the standard positive cone $(\mathbb{Z}^+)^{[X]}$, T is an algebraic antichain in G, and $E := \Pi(G, T)$ is the corresponding effect algebra. The perspectivity class corresponding to $f \in G^+[0, T]$ is $\pi(f)$

$\in E$, the unit in E is given by $u := \pi(t)$ for all $t \in T$, and the zero in E is given by $0 = \pi(0)$. If $x \in X$, then $\chi_x := \chi_{\{x\}}$ denotes the characteristic set function of the set $\{x\}$. Assuming that $\chi_x \in G^+[0, T]$, we define $\pi_x := \pi(\chi_x)$ for all $x \in X$. Also, we choose and fix a multiplicity function $s \in T$, define $T - s := \{t - s | t \in T\}$, and recall that $\langle T - s \rangle$ is the subgroup of G generated by $T - s$. Finally, we assume that $\mathcal{G}$ is an Abelian group and ξ: $G \to \mathcal{G}$ is a group epimorphism with $\ker(\xi) = \langle T - s \rangle$.

Evidently, $x \in X$ implies that $\pi_x \neq 0$ and $\{\pi_x | x \in X\}$ is a set of generators for the effect algebra $E = \Pi(G, T)$. Also, if $t, s' \in T$, then $t - s = (t - s') - (s - s')$, so the subgroup $\langle T - s \rangle$ of G and the quotient group $G/\langle T - s \rangle$ are independent of the choice of s. If desired, one can take the group $\mathcal{G}$ to be the quotient group $G/\langle T - s \rangle$ and ξ to be the natural epimorphism ξ: $G \to G/\langle T - s \rangle$. In any case, $\mathcal{G}$ is isomorphic to $G/\langle T - s \rangle$.

Lemma 5.2. Let $f, g \in G^+[0, T]$. Then:
(i) $f \sim g \Leftrightarrow \pi(f) = \pi(g)$.
(ii) $\pi(f) = u \Leftrightarrow f \in T$.
(iii) $\pi(f) = 0 \Leftrightarrow f = 0$.
(iv) $f \perp g \Leftrightarrow \pi(f) \perp \pi(g)$.
(v) $f \perp g \Rightarrow \pi(f + g) = \pi(f) \oplus \pi(g)$.
(vi) $\pi(f) = \oplus_{x \in X} f(x)\pi_x$.
(vii) $\pi(f) = \pi(g) \Rightarrow \xi(f) = \xi(g)$.

Proof. Parts (i)–(v) are obvious. Since $f = \Sigma_{x \in X} f(x)\chi_x$ and there are only finitely many nonzero terms in the sum, Part (vi) follows from Part (v) and induction. To prove (vii), suppose $\pi(f) = \pi(g)$ so that $f \sim g$ and $\exists h \in G^+[0, T]$ such that $f + h, g + h \in T$. Then,

$$f - g = [(f + h) - s] - [(g + h) - s] \in \langle T - s \rangle = \ker(\xi)$$

so $\xi(f) = \xi(g)$. ∎

If A is an Abelian group, a mapping ϕ: $E \to A$ is called an *A-valued measure* on E iff, for $p, q \in E$ with $p \perp q$, $\phi(p \oplus q) = \phi(p) + \phi(q)$.

Theorem 5.3. There is a unique mapping γ: $E \to \mathcal{G}$ such that:
(i) $f \in G^+[0, T] \Rightarrow \gamma(\pi(f)) = \xi(f)$.
Furthermore:
(ii) γ: $E \to \mathcal{G}$ is a $\mathcal{G}$-valued measure.
(iii) $t \in T \Rightarrow \xi(t) = \gamma(u)$.
(iv) $f \in G \Rightarrow \xi(f) = \Sigma_{x \in X} f(x)\gamma(\pi_x)$.
(v) $\xi(G^+[0, T]) = \gamma(E)$.
(vi) $\xi(G^+) = \mathrm{ssg}(\gamma(E))$.
(vii) $\langle \gamma(E) \rangle = \langle \{\gamma(\pi_x) | x \in X\} \rangle = \mathcal{G}$.

Proof. For $p \in E$, define $\gamma(p) := \xi(f)$, where f is any element in $G^+[0, T]$ such that $\pi(f) = p$. By Part (vii) of Lemma 5.2, $\gamma: E \to \mathcal{G}$ is well defined. Obviously (i) holds and γ is uniquely determined by (i). To prove (ii), suppose $p, q \in E$ with $p \perp q$ and select $f, g \in G^+[0, T]$ with $p = \pi(f)$ and $q = \pi(g)$. Then, by Parts (iv) and (v) of Lemma 5.2, $f + g \in G^+[0, T]$ and $p \oplus q = \pi(f + g)$, whence,

$$\gamma(p \oplus q) = \xi(f + g) = \xi(f) + \xi(g) = \gamma(p) + \gamma(q)$$

Part (iii) follows immediately from Part (ii) of Lemma 5.2.

For $x \in X$, $\chi_x \in G^+[0, T]$, and it follows from (i) that $\gamma(\pi_x) = \xi(\chi_x)$. But, for $f \in G$, $f = \Sigma_{x \in X} f(x)\chi_x$, so $\xi(f) = \Sigma_{x \in X} f(x)\xi(\chi_x) = \Sigma_{x \in X} f(x)\gamma(\pi_x)$, proving (iv). Part (v) follows immediately from (i) and the fact that $\pi(G^+[0, T]) = E$. Since T is generative in G, we have $G^+ = \text{ssg}(G^+[0, T])$ so in view of (v),

$$\xi(G^+) = \xi(\text{ssg}(G^+[0, T])) = \text{ssg}(\xi(G^+[0, T])) = \text{ssg}(\gamma(E))$$

proving Part (vi).

Since $G = G^+ - G^+$ and $\xi: G \to \mathcal{G}$ is surjective, Part (vi) implies that $\langle\gamma(E)\rangle = \mathcal{G}$. Also, for $p \in E$, $\exists f \in G^+[0, T]$ with $\gamma(p) = \xi(f) = \Sigma_{x \in X} f(x)\gamma(\pi_x)$, and it follows that $\langle\gamma(E)\rangle = \langle\{\gamma(\pi_x) \mid x \in X\}\rangle$, completing the proof of Part (vii). ∎

A *universal group* for an effect algebra F is a pair $(\mathcal{U}, \lambda)$ consisting of an Abelian group $\mathcal{U}$ and a $\mathcal{U}$-valued measure $\lambda: F \to \mathcal{U}$ such that (i) $\langle\lambda(F)\rangle = \mathcal{U}$ and (ii) for every Abelian group A and every A-valued measure $\phi: F \to A$, there is a (necessarily unique) group homomorphism $\phi^*: \mathcal{U} \to A$ such that $\phi = \phi^* \circ \lambda$. By Foulis and Bennett (1994), every effect algebra has a universal group which is unique up to an isomorphism. Because of the uniqueness, we often speak of *the* universal group for an effect algebra.

Theorem 5.4. $(\mathcal{G}, \gamma)$ is the universal group for E.

Proof. By Part (vii) of Theorem 5.3, $\langle\gamma(E)\rangle = \mathcal{G}$. Suppose that $\phi: E \to A$ is an A-valued measure and define the group homomorphism $\phi^\#: G \to A$ by $\phi^\#(f) := \Sigma_{x \in X} f(x)\phi(\pi_x)$ for all $f \in G = \mathbb{Z}^{[X]}$. Then, for all $f \in G^+[0, T]$, Part (vi) of Lemma 5.2 implies that

$$\phi(\pi(f)) = \phi\left(\bigoplus_{x \in X} f(x)\pi_x\right) = \sum_{x \in X} f(x)\phi(\pi_x) = \phi^\#(f)$$

In particular, if $t \in T$, then $\phi^\#(t) = \phi(\pi(t)) = \phi(u)$, whence $\phi^\#(t) = \phi^\#(s) = \phi(u)$, so $t - s \in \ker(\phi^\#)$. Therefore, $\ker(\xi) = \langle T - s\rangle \subseteq \ker(\phi^\#)$, so there is a group homomorphism $\phi^*: \mathcal{G} \to A$ such that $\phi^* \circ \xi = \phi^\#$. Thus, for all $f \in G^+[0, T]$,

$$\phi(\pi(f)) = \phi^\#(f) = \phi^*(\xi(f)) = \phi^*(\gamma(\pi(f)))$$

by Part (i) of Theorem 5.3. Therefore, by the surjectivity of $\pi: G^+[0, T] \to E$, we have $\phi = \phi^* \circ \gamma$. ∎

The mapping $\gamma: E \to \mathcal{G}$ is called the *universal* (*$\mathcal{G}$-valued*) *measure* for E and the mapping $\xi: \mathbb{Z}^{[X]} \to \mathcal{G}$ is called the *canonical epimorphism*.

An A-valued measure $\phi: E \to A$ is said to be *positive* iff, for all $p \in E$, $\phi(p) = 0 \Rightarrow p = 0$.

Lemma 5.5. The following conditions are mutually equivalent:

(i) $\gamma: E \to \mathcal{G}$ is positive.

(ii) For every $0 \neq p \in E$ there exists an Abelian group A and an A-valued measure $\phi: E \to A$ such that $\phi(p) \neq 0$.

(iii) $\ker(\xi) \cap G^+[0, T] = \{0\}$.

Proof. (i) $\Rightarrow$ (ii) is obvious. Assume (ii) and suppose that $p \in E$ with $\gamma(p) = 0$. If $p \neq 0$, there is an A-valued measure $\phi: E \to A$ with $\phi(p) \neq 0$. Since $(\mathcal{G}, \gamma)$ is the universal group for E, there is a group homomorphism $\phi^*: \mathcal{G} \to A$ such that $\phi = \phi^* \circ \gamma$. Therefore, $\phi(p) = \phi^*(\gamma(p)) = \phi^*(0) = 0$, contradicting $\phi(p) \neq 0$, and proving that (ii) $\Rightarrow$ (i). To prove (i) $\Rightarrow$ (iii), assume (i) and suppose $f \in \ker(\xi) \cap G^+[0, T]$. Then $0 = \xi(f) = \gamma(\pi(f))$, and the positivity of γ implies $\pi(f) = 0$ so $f = 0$. Conversely, assume (iii) and suppose $p \in E$ with $\gamma(p) = 0$. Select $f \in G^+[0, T]$ with $\pi(f) = p$. Then $0 = \gamma(\pi(f)) = \xi(f)$, so $f \in \ker(\xi) \cap G^+[0, T] = \{0\}$, and it follows that $p = \pi(0) = 0$. Thus, (iii) $\Rightarrow$ (i). ∎

We now consider the problem of partially ordering the universal group $\mathcal{G}$ in such a way that $\gamma(E) \subseteq \mathcal{G}^+$, that is, $\mathrm{ssg}(\gamma(E)) \subseteq \mathcal{G}^+$. If we can do this at all, we might as well take $\mathcal{G}^+ = \mathrm{ssg}(\gamma(E))$.

Lemma 5.6. $\mathcal{G}$ can be organized into a partially ordered Abelian group with positive cone $\mathcal{G}^+ = \mathrm{ssg}(\gamma(E))$ iff $\langle T - s \rangle = \ker(\xi)$ is an order-convex subgroup of G. If $\mathcal{G}$ is partially ordered by the positive cone $\mathcal{G}^+ = \mathrm{ssg}(\gamma(E))$, then:

(i) $\xi: G \to \mathcal{G}$ is an order-preserving group epimorphism.

(ii) $\gamma(E) \subseteq \mathcal{G}^+[0, \gamma(u)]$.

(iii) $\mathcal{G}$ is directed, $\gamma(u)$ is a generative order-unit in $\mathcal{G}$, and $\mathcal{G}^+[0, \gamma(u)]$ is an interval effect algebra.

(iv) If $a \in \mathcal{G}$, then $a \in \mathcal{G}^+[0, \gamma(u)] \Leftrightarrow \exists t \in G^+$, $\xi(t) = \gamma(u)$, and $\exists f \in G^+$ with $f \leq t$ and $a = \xi(f)$.

(v) γ is positive $\Leftrightarrow \gamma(\pi_x) \neq 0$ for all $x \in X$.

(vi) γ is positive $\Leftrightarrow \ker(\xi) \cap G^+ = \{0\}$.

Proof. By Part (vi) of Theorem 5.3, $\xi(G^+) = \text{ssg}(\gamma(E))$ and, by Theorem 1.3, we can partially order $\mathcal{G}$ with $\mathcal{G}^+ = \xi(G^+)$ iff $\ker(\xi)$ is order convex in G. Assume that $\mathcal{G}$ is partially ordered by $\mathcal{G}^+ = \text{ssg}(\gamma(E)) = \xi(G^+)$. Then Part (i) is obvious. To prove Part (ii), suppose $p \in E$. Then $\exists f \in G^+[0, T]$ with $p = \pi(f)$ and $\exists t \in T$ with $0 \leq f \leq t$. Therefore by Part (i), $0 \leq \xi(f) \leq \xi(t)$, whence $0 \leq \gamma(\pi(f)) \leq \gamma(\pi(t))$ by Part (i) of Theorem 5.3, so $0 \leq \gamma(p) \leq \gamma(u)$, proving Part (ii). In Part (iii), the fact that $\mathcal{G}$ is directed follows from the facts that G is directed, $\mathcal{G}^+ = \xi(G^+)$, and $\xi(G) = \mathcal{G}$. The fact that $\gamma(u)$ is generative follows from Part (ii) and the fact that $\text{ssg}(\gamma(E)) = \xi(G^+) = \mathcal{G}^+$. Therefore, $\gamma(u)$ is an order-unit by Lemma 1.8.

In (iv), the implication $\Leftarrow$ follows from the fact that ξ is order-preserving. To prove the converse, suppose $0 \leq a \leq \gamma(u)$. Then $\exists f, g \in G^+$ such that $a = \xi(f)$ and $\gamma(u) - a = \xi(g)$. Let $t := f + g$, noting that $\xi(t) = \gamma(u)$, $t \in G^+$, and $f \leq t$.

In (v), the implication $\Leftarrow$ follows from the fact that $\pi_x \neq 0$ for all $x \in X$. To prove the converse, suppose $p \in E$ with $\gamma(p) = 0$ and select $f \in G^+[0, T]$ with $p = \pi(f)$. Then $0 = \gamma(p) = \xi(f) = \Sigma_{x \in X} f(x)\gamma(\pi_x)$. If $\gamma(\pi_x) \neq 0$ for all $x \in X$, then $0 \neq \gamma(\pi_x) \in \mathcal{G}^+$, and, since $f(x) \in \mathbb{Z}^+$ for all $x \in X$, it follows that $f(x) = 0$ for all $x \in X$, whence $p = \pi(0) = 0$.

In (vi), the implication $\Leftarrow$ follows from Lemma 5.5 and the fact that $G^+[0, T] \subseteq G^+$. To prove the converse, suppose γ is positive and let $g \in \ker(\xi) \cap G^+$. Since $G^+[0, T]$ is generative and $g \in G^+$, $\exists f_1, f_2, \ldots, f_n \in G^+[0, T]$ with $g = \Sigma_i f_i$. Each $f_i \leq g$, so the order-convexity of $\ker(\xi)$ implies that $f_i \in \ker(\xi)$ for $i = 1, 2, \ldots, n$. By Lemma 5.5, $\ker(\xi) \cap G^+[0, T] = \{0\}$, so $f_i = 0$ for $i = 1, 2, \ldots, n$, and it follows that $g = 0$. ∎

Definition 5.7. T is *complete* iff for all $t \in G^+$, $\xi(t) = \gamma(u) \Rightarrow t \in T$.

Lemma 5.8. If $\mathcal{G}$ is partially ordered by the positive cone $\mathcal{G}^+ = \text{ssg}(\gamma(E)) = \xi(G^+)$ and T is complete, then $\gamma(E) = \mathcal{G}^+[0, \gamma(u)]$.

Proof. Suppose T is complete and let $a \in \mathcal{G}^+[0, \gamma(u)]$. By Part (iv) of Lemma 5.6, $\exists t \in G^+$, $\xi(t) = \gamma(u)$ and $\exists f \in G^+$ with $f \leq t$ and $a = \xi(f)$. Since T is complete, $t \in T$; hence, $f \in G^+[0, T]$, so, by Part (i) of Theorem 5.3, $a = \xi(f) = \gamma(\pi(f)) \in \gamma(E)$. ∎

Lemma 5.9. If $\mathcal{G}$ is partially ordered by $\mathcal{G}^+ = \text{ssg}(\gamma(E)) = \xi(G^+)$, then the following conditions are mutually equivalent:

(i) $\gamma\colon E \to \mathcal{G}^+[0, \gamma(u)]$ is an effect algebra isomorphism.

(ii) E is an interval effect algebra.

(iii) For all $p, q \in E$, $\gamma(p) \leq \gamma(q) \Rightarrow p \leq q$.

Proof. By Part (iii) of Lemma 5.6, $\mathcal{G}^+[0, \gamma(u)]$ is an interval effect algebra, so (i) $\Rightarrow$ (ii) holds. To prove (ii) $\Rightarrow$ (iii), suppose E is an interval effect algebra. Then there is a directed, partially ordered Abelian group A, a generative order unit $v \in A$, and an effect-algebra isomorphism ϕ: $E \to A^+[0, v]$. In particular, ϕ is an A-valued measure, so there is a group homomorphism ϕ^*: $\mathcal{G} \to A$ such that $\phi = \phi^* \circ \gamma$. Thus,

$$\phi^*(\mathcal{G}^+) = \phi^*(\text{ssg}(\gamma(E))) = \text{ssg}(\phi^*(\gamma(E))) = \text{ssg}(\phi(E))$$
$$\subseteq \text{ssg}(A^+) = A^+$$

so ϕ^* is order-preserving. If $p, q \in E$ with $\gamma(p) \leq \gamma(q)$, then $\phi^*(\gamma(p)) \leq \phi^*(\gamma(q))$, that is, $\phi(p) \leq \phi(q)$, so $p \leq q$, completing the proof of (ii) $\Rightarrow$ (iii).

If (iii) holds, then $\gamma(E)$ is a sub-effect algebra of the interval effect algebra $\mathcal{G}^+[0, \gamma(u)]$ and E is isomorphic to $\gamma(E)$ under the isomorphism γ. Therefore, by Corollary 2.6 in Bennett and Foulis (n.d.), E is an interval effect algebra, and we have (iii) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (i) follows from Theorem 4.2 in Bennett and Foulis (n.d.-). ■

Corollary 5.10. If E is an interval effect algebra, then T is complete.

Proof. Suppose E is an interval effect algebra and let $t \in G^+$ with $\xi(t) = \Sigma_{x \in X}\, t(x)\gamma(\pi_x) = \gamma(u)$. Then $\oplus_{x \in X}\, t(x)\gamma(\pi_x) = \gamma(u)$ in $\mathcal{G}^+[0, \gamma(u)]$, so $\gamma(\oplus_{x \in X}\, t(x)\chi_x) = \gamma(u)$, whence $\oplus_{x \in X}\, t(x)\pi_x = u$ in E, so $t \in T$. ■

Theorem 5.11. E is an interval effect algebra iff the following conditions hold:

(i) $\ker(\xi) \cap G^+ = \{0\}$, so $\ker(\xi)$ is trivially order-convex and $\mathcal{G}$ is partially ordered by the positive cone $\mathcal{G}^+ = \text{ssg}(\gamma(E)) = \xi(G^+)$.

(ii) For all $p, q \in E$, $\gamma(p) \leq \gamma(q) \Rightarrow p \leq q$.

Proof. Suppose (i) and (ii) hold. By (i), $\mathcal{G}$ is partially ordered with positive cone $\mathcal{G}^+ = \text{ssg}(\gamma(E))$; hence, by (ii) and Lemma 5.9, E is an interval effect algebra.

Conversely, suppose E is an interval effect algebra. By Theorem 4.2 in Bennett and Foulis (n.d.), $\mathcal{G}$ is partially ordered by the cone $\mathcal{G}^+ = \text{ssg}(\gamma(E))$, $\mathcal{G}^+[0, \gamma(u)]$ is an interval effect algebra, and γ: $E \to \mathcal{G}^+[0, \gamma(u)]$ is an effect-algebra isomorphism. Thus, Condition (ii) holds and γ is positive, so $\ker(\xi) \cap G^+ = \{0\}$ by Part (vi) of Lemma 5.6, proving Condition (i). ■

6. COMPUTATIONAL ALGORITHMS FOR FINITE EFFECT ALGEBRAS

The results obtained above can be used to formulate computational algorithms for dealing with simplicial test groups and finite effect algebras, and the resulting algorithms are easily implemented on a computer. We devote the present section to a brief outline of a basis for the development of some of these algorithms.

If $X = \{x_1, x_2, \ldots, x_m\}$ is a finite set, an element f in the simplicial group $\mathbb{Z}^X$ can be represented in the usual way by an m-vector $(f_1, f_2, \ldots, f_m) := (f(x_1), f(x_2), \ldots, f(x_m))$ in $\mathbb{Z}^m$. By Lemma 1.11, if $(\mathbb{Z}^m, S)$ is a simplicial test group, then the antichain S is necessarily finite and Theorem 4.6 provides an effective algorithm for checking whether S is prealgebraic and for computing its algebraic closure $S^{\sim}$ if it is.

Example 6.1. For the simplicial group $\mathbb{Z}^3$, let

$$S = \{(0, 4, 0), (3, 1, 0), (2, 0, 2), (0, 1, 3)\}$$

Then $(\mathbb{Z}^3, S)$ is a simplicial test group, but it is not algebraic because $(2, 0, 2) \leq (3, 1, 0) + (0, 1, 3)$ and $(1, 2, 1) = (3, 1, 0) + (0, 1, 3) - (2, 0, 2) \notin T$. Here $\mathfrak{D}(S) = S \cup \{(1, 2, 1)\}$ and $\mathfrak{D}(S)$ is an antichain. Furthermore, $\mathfrak{D}(\mathfrak{D}(S)) = \mathfrak{D}(S)$, so S is prealgebraic and $T := S^{\sim} = S \cup \{(1, 2, 1)\}$ is algebraic.

If $(\mathbb{Z}^m, T)$ is an algebraic simplicial test group, then the T-events in $(\mathbb{Z}^+)^m[0, T]$ can be computed by listing, for each $t \in T$, all vectors $v \in (\mathbb{Z}^+)^m$ with $v \leq t$, and then removing all repetitions from the list. For instance, in Example 6.1, one finds that there are 25 T-events in $(\mathbb{Z}^+)^3[0, T]$.

Let T be an algebraic antichain of tests in $\mathbb{Z}^m$. If $e \in \mathscr{E} := (\mathbb{Z}^+)^m[0, T]$, the set $\pi(e)'$ of all supplements of e in $(\mathbb{Z}^+)^m[0, T]$ is calculated by listing all vectors of the form $t - e$ for $t \in T$ and discarding those that do not belong to $(\mathbb{Z}^+)^m$. The set $\pi(e)'$ is a perspectivity class in the effect algebra $\Pi(\mathbb{Z}^m, T)$, and every perspectivity class arises in this way. By choosing any $f \in \pi(e)'$ and calculating $\pi(f)'$, one finds the orthosupplement $\pi(e)$ of $\pi(f)'$ in $\Pi(\mathbb{Z}^m, T)$. One now removes the vectors in the classes $\pi(e)$ and $\pi(e)'$ from $\mathscr{E}$ and proceeds iteratively to calculate the elements of $\Pi(\mathbb{Z}^m, T)$ in orthosupplementary pairs. Of course, one must check to make sure that the classes in such a pair are distinct; if they are not, one has found a "*half-element*" in $\Pi(\mathbb{Z}^m, T)$, i.e., an element $\pi(e)$ such that $\pi(e) \oplus \pi(e)$ is the unit in $\Pi(\mathbb{Z}^m, T)$.

Example 6.2. For the simplicial group $\mathbb{Z}^3$ with the algebraic antichain

$$T = \{(0, 4, 0), (3, 1, 0), (2, 0, 2), (0, 1, 3), (1, 2, 1)\}$$

obtained in Example 6.1, there are 11 perspectivity classes in the effect algebra $\Pi(\mathbb{Z}^3, T)$. The classes in orthosupplementary pairs are: $\pi(0, 0, 0)$ and $\pi(0, 4, 0)$, $\pi(1, 0, 0)$ and $\pi(2, 1, 0)$, $\pi(0, 1, 0)$ and $\pi(1, 1, 1)$, $\pi(0, 0, 1)$ and $\pi(2, 0, 1)$, $\pi(2, 0, 0)$ and $\pi(0, 0, 2)$, and the half-element $\pi(0, 2, 0)$ (which is its own orthosupplement). Note that $\pi(1, 0, 0)$, $\pi(0, 1, 0)$, and $\pi(0, 0, 1)$ are the three atoms in $\Pi(\mathbb{Z}^3, T)$.

Once the elements of $\Pi(\mathbb{Z}^m, T)$ are identified, it becomes possible to work out the structure of $\Pi(\mathbb{Z}^m, T)$ as an effect algebra. For instance, in Example 6.2, it is not difficult to see that $\Pi(\mathbb{Z}^3, T)$ is the polychain $C_{4,3}$ (Foulis *et al.*, 1994, Example 4.7).

If $(\mathbb{Z}^m, T)$ is an algebraic simplicial test group and E is isomorphic to $\Pi(\mathbb{Z}^m, T)$, then the universal group $\mathcal{G}$ of E can be calculated by standard group-theoretic algorithms (Hungerford, 1974, pp. 343–345), since a presentation of $\mathcal{G}$ is given by generators corresponding to the standard free basis $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$ for $\mathbb{Z}^m$ with relations corresponding to $t - s = 0$ for $t \in T$ and s fixed in T. Thus $\mathcal{G}$ is a finitely-generated Abelian group and hence it is isomorphic to a Cartesian product of a free Abelian group $\mathbb{Z}^r$ and a finite Abelian group $\mathcal{J}$ which itself is a Cartesian product of finite cyclic groups.

Example 6.3. For the algebraic simplicial test group $(\mathbb{Z}^3, T)$ in Examples 6.1 and 6.2, the generators $\gamma_1 := \gamma(1, 0, 0)$, $\gamma_2 := \gamma(0, 1, 0)$, and $\gamma_3 := \gamma(0, 0, 1)$ of the universal group $\mathcal{G}$ are subject to the relations $3\gamma_1 - 3\gamma_2 = 0$, $2\gamma_1 - 4\gamma_2 + 2\gamma_3 = 0$, $\gamma_1 - 4\gamma_2 + 3\gamma_3 = 0$, and $\gamma_1 - 2\gamma_2 + \gamma_3 = 0$ corresponding to $(3, 1, 0) - (0, 4, 0)$, $(2, 0, 2) - (0, 4, 0)$, $(1, 0, 3) - (0, 4, 0)$, and $(1, 2, 1) - (0, 4, 0)$, respectively. By standard group-theoretic methods, we find that $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}_3$ with $\gamma_1 = (1, 2)$, $\gamma_2 = (1, 1)$, and $\gamma_3 = (1, 0)$. If u is the unit in $E = \Pi(\mathbb{Z}^3, T)$, then $\gamma(u) = (4, 1)$ as can be seen, for instance, by using the fact that $\gamma(u) = 4\gamma(0, 1, 0) = 4\gamma_2 = (4, 4) = (4, 1)$. The canonical epimorphism $\xi\colon \mathbb{Z}^3 \to \mathcal{G} = \mathbb{Z} \times \mathbb{Z}_3$ is given by $\xi(x, y, z) = (x + y + z, \alpha)$, where $\alpha \equiv 2x + y$ modulo 3.

In Example 6.3, $\ker(\xi) = \{(n - 3m, 3m - 2n, n) \mid n, m \in \mathbb{Z}\}$, so $\ker(\xi) \cap (\mathbb{Z}^+)^3 = \{0\}$ and $\mathcal{G}$ is partially ordered by the positive cone $\mathcal{G}^+ = \xi((\mathbb{Z}^+)^3) = (\mathbb{Z}^+ \times \mathbb{Z}_3)\backslash\{(0, 1), (0, 2)\}$. Furthermore, γ satisfies Condition (ii) in Theorem 5.11, so $\Pi(\mathbb{Z}^3, T)$ is an interval effect algebra isomorphic to $\mathcal{G}^+[(0, 0), (4, 1)]$.

7. PROLEGOMENON TO A THEORY OF QUOTIENTS

In this final section, we briefly sketch some ideas that may be useful in the development of a theory of quotients for test groups, effect algebras, and interval effect algebras.

An *ideal* in an effect algebra E is a subset $I \subseteq E$ such that (i) $0 \in I$, (ii) $p, q \in E, p \le q, q \in I \Rightarrow p \in I$, and (iii) $p, q \in I$ with $p \perp q \Rightarrow p \oplus q \in I$. We consider the problem of formulating an adequate definition of a "quotient" effect algebra E/I. For such a definition, there are a number of desiderata; among the most important of these are perhaps the following:

1. The definition should be compatible with the standard notion of a quotient of an orthomodular lattice by a p-ideal (Kalmbach, 1983). In particular, it should be compatible with the standard construction of a quotient of a Boolean algebra by a Boolean ideal.
2. If E and F are effect algebras and I is the ideal in $E \times F$ consisting of all ordered pairs $(0, f)$ with $f \in F$, then $(E \times F)/I$ should be isomorphic to E.
3. There should be a "natural morphism" $E \to E/I$ reducing in the case of an orthomodular lattice or a Boolean algebra to the usual natural homomorphism.
4. If ω is a probability measure (or state) on E (Foulis *et al.*, 1994) and $I = \{p \in E \mid \omega(p) = 0\}$, then ω should induce in a natural way a probability measure ω^* on E/I that is strictly positive in the sense that $0 < \omega^*(q)$ for all $0 \neq q \in E/I$.
5. Quotients should interact with tensor products (Bennett and Foulis, 1993; Dvurečenskij and Pulmannová, 1994a; Foulis *et al.*, 1994; Dvurečenskij, 1995; Gudder, 1995) more or less as they do in the category of modules over commutative rings (e.g., Bourbaki, 1958, Section 1, Proposition 6).

Because of the close connection between test groups and effect algebras, it seems appropriate to begin by formulating a definition of ideals and quotients for test groups.

Definition 7.1. Let (G, T) be a test group, let H be a directed order-convex subgroup of G, let $\eta\colon G \to G/H$ be the natural group epimorphism, and consider G/H to be partially ordered with $(G/H)^+ := \eta(G^+)$ as its positive cone. We say that H is a *T-ideal* in G iff $\eta(T)$ is an antichain in G/H.

If H is a T-ideal in G, then $H = \langle H \cap G^+[0, T]\rangle$ in Lemma 1.9, so H is actually generated by a set of T-events.

Lemma 7.2. If (G, T) is a test group, H is a T-ideal in G, and $\eta\colon G \to G/H$ is the natural group epimorphism, then $(G/H, \eta(T))$ is a test group and $\eta(G^+[0, T]) \subseteq (G/H)^+[0, \eta(T)]$.

Proof. Because G is directed and η is surjective and order-preserving, G/H is directed. Also, since η is order-preserving, $\eta(G^+[0, T]) \subseteq (G/H)^+[0, \eta(T)]$, so

$$\begin{aligned}(G/H)^+ = \eta(G^+) = \eta(\mathrm{ssg}(G^+[0, T]) &\subseteq \mathrm{ssg}(\eta(G^+[0, T]))\\ &\subseteq \mathrm{ssg}((G/H)^+[0, \eta(T)])\end{aligned}$$

and $(G/H)^+[0, \eta(T)]$ is generative. ■

One might expect that $\eta(G^+[0, T]) = (G/H)^+[0, \eta(T)]$ would hold in Lemma 7.2, but suitable counterexamples show that this need not be the case. Also, even if (G, T) is algebraic, $(G/H, \eta(T))$ need not be algebraic.

Definition 7.3. If (G, T) is a test group, H is a T-ideal in G, $\eta: G \to G/H$ is the natural group epimorphism, and the test group $(G/H, \eta(T))$ is prealgebraic, we define the *quotient test group* $(G, T)/H := (G/H, \eta(T)^{\sim})$.

A connection between ideals in effect algebras and T-ideals is provided by the following theorem.

Theorem 7.4. Let I be an ideal in the effect algebra E, let $X \subseteq E \setminus \{0\}$ be a set of generators for E, let $Y := X \setminus I$, and let $(\mathbb{Z}^{[X]}, T)$ be the multiplicity group for E with respect to X. Then $H := \{f \in \mathbb{Z}^{[X]} | f(y) = 0$ for all $y \in Y\}$ is a T-ideal in $\mathbb{Z}^{[X]}$.

Proof. By Theorem 1.10, H is an order-convex directed subgroup of $\mathbb{Z}^{[X]}$. If $f \in \mathbb{Z}^{[X]}$, let f_Y be the restriction of f to Y, noting that $f \mapsto f_Y$ is a group epimorphism with kernel H from $\mathbb{Z}^{[X]}$ onto $\mathbb{Z}^{[Y]}$ and $(\mathbb{Z}^+)^{[Y]}$ is the image of $(\mathbb{Z}^+)^{[X]}$ under $f \mapsto f_Y$. Therefore, we can identify $\mathbb{Z}^{[X]}/H$ with $\mathbb{Z}^{[Y]}$, η with $f \mapsto f_Y$, and $\eta(T)$ with $T_Y := \{t_Y | t \in T\}$. Suppose $s, t \in T$ with $s_Y \leq t_Y$ and let $p := \oplus_{x \in I} s(x)x$, $q := \oplus_{x \in I} t(x)x$, $a := \oplus_{y \in Y} s(y)y$, and $b := \oplus_{y \in Y} (t(y) - s(y))y$. Then $p \oplus a = q \oplus a \oplus b = u$ and, since I is an ideal, $p, q \in I$. By the cancellation law in E, $p = q \oplus b$, so $b \leq q \in I$, and it follows that $b \in I$. If $y \in Y$ and $s(y) < t(y)$, then $y \leq (t(y) - s(y))y \leq b \in I$, so $y \in I$, contradicting the definition of Y. Therefore, $y \in Y \Rightarrow s(y) = t(y)$, so $s_Y = t_Y$ and T_Y is an antichain in $\mathbb{Z}^{[Y]}$. ∎

Availing ourselves of the ideas in the proof of Theorem 7.4, we formulate the following general definition of the quotient of an effect algebra E by an ideal I.

Definition 7.5. Let $X \subseteq E \setminus \{0\}$ be a set of generators for the effect algebra E and let I be an ideal in E. Let $Y := X \setminus I$; for each $f \in \mathbb{Z}^{[X]}$, let f_Y be the restriction of f to Y. If T is the set of multiplicity functions for E with respect to X, let $T_Y := \{t_Y | t \in T\}$. If the multiplicity group $(\mathbb{Z}^{[Y]}, T_Y)$ is prealgebraic, we define the *quotient effect algebra* $E/I := \Pi(\mathbb{Z}^{[Y]}, T_Y^{\sim})$.

Example 7.6. Let $T \subseteq \mathbb{Z}^5$ be the algebraic antichain consisting of (0, 1, 0, 1, 1), (1, 1, 0, 0, 0), and (0, 0, 1, 1, 0). Then $E := \Pi(\mathbb{Z}^5, T)$ consists of eight perspectivity classes as follows: three atoms $\pi(0, 1, 0, 0, 0)$, $\pi(0, 0, 0, 1, 0)$, $\pi(0, 0, 0, 0, 1)$; their respective orthosupplements $\pi(1, 0, 0, 0, 0)$, $\pi(0, 0, 1, 0, 0)$, $\pi(0, 1, 0, 1, 0)$; and the zero and unit. In fact, E is the eight-element Boolean algebra. Let I be the ideal in E consisting of $\pi(0, 0, 0, 0, 0)$ and $\pi(0, 0, 0, 0, 1)$. To pass to the test space $(\mathbb{Z}^4, T_Y)$ we remove the last component from each vector in T to obtain the antichain T_Y consisting of (0, 1, 0, 1), (1, 1, 0, 0), and (0, 0, 1, 1) in $\mathbb{Z}^4$. But T_Y is not algebraic, since

$$(1, 1, 0, 0) + (0, 0, 1, 1) - (0, 1, 0, 1) = (1, 0, 1, 0) \notin T_Y$$

However, $\mathfrak{D}(T_Y) = T_Y \cup \{(1, 0, 1, 0)\}$ is algebraic, so T_Y is prealgebraic with $\widetilde{T_Y} = \mathfrak{D}(T_Y)$. The quotient $E/I := \Pi(\mathbb{Z}^4, \widetilde{T_Y})$ is the four-element Boolean algebra.

The following alternative definition of quotient suggests itself in the category of interval effect algebras.

Definition 7.7. Let $\mathcal{G}$ be the universal group of the interval effect algebra E with unit u, and identify E with $\mathcal{G}^+[0, u]$. If I is an ideal in E, let $H =$ ocs(I) and let η: $\mathcal{G} \to \mathcal{G}/H$ be the natural group epimorphism. Since H is order-convex, $\mathcal{G}/H$ is partially ordered by the positive cone $(\mathcal{G}/H)^+ := \eta(\mathcal{G}^+)$. Define the *quotient* E/I in the category of interval effect algebras to be $(\mathcal{G}/H)^+[0, \eta(u)]$.

Example 7.8. We reconsider Example 7.6 from the viewpoint of Definition 7.7. The eight-element Boolean algebra has $\mathbb{Z}^3$ as its universal group and may be identified with $E := (\mathbb{Z}^+)^3[(0, 0, 0), (1, 1, 1)]$. Let I be the ideal in E consisting of (0, 0, 0) and (0, 0, 1). Then $H :=$ ocs(I) is the cyclic subgroup $\{(0, 0, z) \mid z \in \mathbb{Z}\}$ generated by (0, 0, 1) and the quotient group $\mathbb{Z}^3/H$ may be identified with $\mathbb{Z}^2$ in such a way that the natural epimorphism η: $\mathbb{Z}^3 \to \mathbb{Z}^3/H$ is identified with the mapping $(x, y, z) \mapsto (x, y)$. Thus E/I may be identified with $(\mathbb{Z}^+)^2[(0, 0), (1, 1)]$, which is the four-element Boolean algebra.

In a forthcoming paper, we study the articulation among the three definitions of quotient given above as well as the question of the satisfaction of desiderata 1–5.

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